

Research *Summery* (it's a pun)

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1 Research goal

The driving goal of this research project with Dr. Yiannis Loizides is to be able to calculate integrals of the following form:

$$\int_{\mathcal{C}_{\bar{\theta}}^{S^1}} e^{\int_0^1 \bar{y}'(t) d\bar{x}(t)} d\mu_{\bar{\theta}}^{S^1}(\bar{x}). \quad (1)$$

Mathematicians care about integrals of this form, because Frenkel [3] showed that for certain groups, given some representation, one can get a notable formula for its character, which is roughly the above goal integral in product with another factor. Let's take a look at the pieces that make up this integral.

1.1 The space(s)

The space over which we are integrating ($\mathcal{C}_{\bar{\theta}}^{S^1}$) is the space of continuous paths on the circle S^1 beginning at $\bar{0}$ and ending at some $\bar{\theta}$. Symbolically, this is,

$$\mathcal{C}_{\bar{\theta}}^{S^1} = \{\bar{x} : [0, 1] \rightarrow S^1 \mid \bar{x}(t) \text{ is continuous, } \bar{x}(0) = \bar{0}, \bar{x}(1) = \bar{\theta}\}.$$

Throughout the text, we also consider the space with no specified endpoint,

$$\mathcal{C}^{S^1} = \{\bar{x} : [0, 1] \rightarrow S^1 \mid \bar{x}(t) \text{ is continuous, } \bar{x}(0) = \bar{0}\}.$$

We also refer to the space of continuous paths in \mathbb{R} beginning at 0,

$$\mathcal{C}^{\mathbb{R}} = \{x : [0, 1] \rightarrow \mathbb{R} \mid x(t) \text{ is continuous, } x(0) = 0\},$$

along with those paths in \mathbb{R} which have specified endpoints θ ,

$$\mathcal{C}_{\theta}^{\mathbb{R}} = \{x : [0, 1] \rightarrow \mathbb{R} \mid x(t) \text{ is continuous, } x(0) = 0, x(1) = \theta\}.$$

Notationally, a bar over an element ($\bar{0}, \bar{\theta}, \bar{x}$, etc.) indicates this element is in S^1 or \mathcal{C}^{S^1} , while the same element without a bar ($0, \theta, x$, etc.) is in \mathbb{R} or $\mathcal{C}^{\mathbb{R}}$.

1.2 The Riemann-Stieltjes integral

$\int_0^1 y'(t) dx(t)$ is called the Riemann-Stieltjes integral [1]. This is defined using a Riemann-Stieltjes sum which involves a partition of the interval over which the integral is defined — in this case, $[0, 1]$. Suppose $P = \{a_0, a_1, a_2, \dots, a_{n-1}, a_n\}$ is a partition of the interval $[0, 1]$, in which $a_0 = 0$ and $a_n = 1$. Let t_k be a point inside the subinterval $[a_{k-1}, a_k]$. Then, a *Riemann-Stieltjes sum of y' with respect to x and P* is

$$S(P, y', x) = \sum_{k=1}^n y'(t_k)(x(a_k) - x(a_{k-1})).$$

Notice that this sum is dependent on the choice of t_k , and it differs from a traditional Riemann sum, which would replace $x(a_k) - x(a_{k-1})$ with $a_k - a_{k-1}$. In other words, the Riemann sum uses the change over consecutive points of the partition, while the Riemann-Stieltjes sum uses the change over the output of some function $x(t)$ whose inputs are these consecutive points in the partition.

The *Riemann-Stieltjes integral* is then defined to be

$$\int_0^1 y'(t)dx(t) = \lim_{n \rightarrow \infty} \frac{1}{n} S(P, y', x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n y'(t_k)(x(a_k) - x(a_{k-1})),$$

and if this limit exists, one says that y' is *Riemann-integrable with respect to x on $[0, 1]$* .

1.3 The measure

We must also define the measures $\mu_{\bar{\theta}}^{S^1}$ in our goal integral. Consider the map $\pi : \mathcal{C}^{S^1} \rightarrow S^1$ defined $\pi(\bar{x}) = \bar{x}(1)$, sending a path to its endpoint. In short, \mathcal{C}^{S^1} is already equipped with a measure ν , which is the pushforward of the Wiener measure from $\mathcal{C}^{\mathbb{R}}$. This measure will be formally explored shortly; for now, we just care that \mathcal{C}^{S^1} is equipped with a measure. We can then pushforward this measure ν on \mathcal{C}^{S^1} to S^1 using our map π to get the measure $\pi_*\nu = \nu\pi^{-1}$ on S^1 . Terence Tao's disintegration theorem [4] says that there exist measures on each fiber of the map π , i.e., on $\pi^{-1}(\bar{\theta})$ for each $\bar{\theta} \in S^1$. Notice that $\pi^{-1}(\bar{\theta}) = \mathcal{C}_{\bar{\theta}}^{S^1}$, precisely the continuous paths in S^1 beginning at $\bar{0}$ and ending at $\bar{\theta}$. These measures on $\mathcal{C}_{\bar{\theta}}^{S^1}$ given by disintegration theorem are denoted $\mu_{\bar{\theta}}^{S^1}$ for each fiber $\pi^{-1}(\bar{\theta})$.

2 Wiener measure

2.1 Definitions

Now, we further define and examine Wiener measure [3]. Though this measure can be generalized to paths in \mathbb{R}^n , we focus on paths in \mathbb{R} . Wiener measure is defined on cylinder sets in $\mathcal{C}^{\mathbb{R}}$. A *cylinder set* is defined for times $0 < t_1 < t_2 < \dots < t_n \leq 1$ and Borel sets $A_1, A_2, \dots, A_n \subseteq \mathbb{R}$ as a set of the form

$$\{x \in \mathcal{C}^{\mathbb{R}} \mid x(t_1) \in A_1, x(t_2) \in A_2, \dots, x(t_n) \in A_n\}.$$

Wiener measure w for a cylinder set K is defined in the following way:

$$w(K) = \int_{A_1} \int_{A_2} \dots \int_{A_n} u(\Delta x_1, \Delta t_1) u(\Delta x_2, \Delta t_2) \dots u(\Delta x_n, \Delta t_n) dx_n \dots dx_2 dx_1, \quad (2)$$

where $\Delta x_k = x_k - x_{k-1}$ (the variables of integration), $\Delta t_k = t_k - t_{k-1}$, $t_0 = 0$, $t_n = 1$, and dx is Lebesgue measure in \mathbb{R} . The functions u are Gaussian distributions with mean 0 and variance 1, so

$$u(x, t) = (2\pi t)^{-1/2} e^{-x^2/2t}.$$

We also want to define a measure on $\mathcal{C}_{\bar{\theta}}^{\mathbb{R}}$. For a cylinder set $K \subseteq \mathcal{C}_{\bar{\theta}}^{\mathbb{R}}$, we define the *conditional Wiener measure* $w_{\bar{\theta}}$ of K as

$$w_{\bar{\theta}}(K) = \int_{A_1} \int_{A_2} \dots \int_{A_{n-1}} u(\Delta x_1, \Delta t_1) u(\Delta x_2, \Delta t_2) \dots u(\Delta x_n, \Delta t_n) dx_{n-1} \dots dx_2 dx_1$$

with the same notation as in Wiener measure with the additional constraint that $x_n = \bar{\theta}$.

Based on these definitions, there is a notable relationship between integrals over $\mathcal{C}^{\mathbb{R}}$ with respect to Wiener measure w and integrals over $\mathcal{C}_{\theta}^{\mathbb{R}}$ with respect to conditional Wiener measure w_{θ} in that, for any function $f \in \mathcal{C}(\mathcal{C}^{\mathbb{R}})$, we have

$$\int_{\mathcal{C}^{\mathbb{R}}} f(x)dw(x) = \int_{\mathbb{R}} \int_{\mathcal{C}_{\theta}^{\mathbb{R}}} f(x)dw_{\theta}(x)d\theta, \quad (3)$$

where $d\theta$ is Lebesgue measure in \mathbb{R} .

2.2 Relating $\mathcal{C}^{\mathbb{R}}$ to \mathcal{C}^{S^1}

We have Wiener measure defined for continuous paths in \mathbb{R} ; our goal is to use a bijection between $\mathcal{C}^{\mathbb{R}}$ and \mathcal{C}^{S^1} to pushforward Wiener measure to \mathcal{C}^{S^1} . Notice that \mathbb{R} is a covering space for S^1 via the covering map $p : \mathbb{R} \rightarrow S^1$ defined $p(r) = (\cos(2\pi r), \sin(2\pi r))$. Further, given some continuous path $\bar{x} : [0, 1] \rightarrow S^1$ with $\bar{x}(0) = \bar{0}$, there exists a unique continuous path $x : [0, 1] \rightarrow \mathbb{R}$ with $x(0) = 0$ and $p(x) = \bar{x}$. This implies the existence of a bijection $g : \mathcal{C}^{\mathbb{R}} \rightarrow \mathcal{C}^{S^1}$, which allows us to pushforward Wiener measure w from $\mathcal{C}^{\mathbb{R}}$ to get a measure $g_*w = wg^{-1}$ on \mathcal{C}^{S^1} . This is the measure on \mathcal{C}^{S^1} that we called ν in Section 1.3.

The next logical step would be to consider what happens if we force our path $\bar{x} \in \mathcal{C}^{S^1}$ to end at a specific point $\bar{\theta}$. In this case, we have that $\mathcal{C}_{\bar{\theta}}^{S^1}$ is in bijection with $\mathcal{C}_{p^{-1}(\bar{\theta})}^{\mathbb{R}}$, continuous paths in \mathbb{R} whose endpoint is in the fiber $p^{-1}(\bar{\theta})$. It is a bit trickier to pushforward a measure from $\mathcal{C}_{p^{-1}(\bar{\theta})}^{\mathbb{R}}$, as this space involves paths with multiple potential endpoints; this is where Terence Tao's disintegration theorem was applied previously (Section 1.3) to equip $\mathcal{C}_{\bar{\theta}}^{S^1}$ with a measure. This relationship allows us to reformulate our research goal integral as

$$\int_{\mathcal{C}_{\bar{\theta}}^{S^1}} e^{\int_0^1 \bar{y}'(t)d\bar{x}(t)} d\mu_{\bar{\theta}}^{S^1}(\bar{x}) = \int_{p^{-1}(\bar{\theta})} \int_{\mathcal{C}_{\theta}^{\mathbb{R}}} e^{\int_0^1 y'(t)dx(t)} dw_{\theta}(x)d\theta \quad (4)$$

in which $\theta \in p^{-1}(\bar{\theta})$, $\bar{x} \in \mathcal{C}_{\bar{\theta}}^{S^1}$ lifts to $x \in \mathcal{C}_{\theta}^{\mathbb{R}}$ via the covering map p , and $\bar{y} \in \mathcal{C}_{\bar{\theta}}^{S^1}$ lifts to $y \in \mathcal{C}_{\theta}^{\mathbb{R}}$ via the covering map p .

3 Relevant formulas and theorems

There are several important theorems and formulas that will help in manipulating and calculating the target integral. They are briefly mentioned here.

3.1 The Cameron-Martin Formula

The general idea of the Cameron-Martin formula is in explaining how Wiener measure changes under a specific type of translation. This relates to the *translational quasi-invariance* of Wiener measure:

Translational Quasi-invariance: Given Wiener measure w on $\mathcal{C}^{\mathbb{R}}$ and a fixed smooth path $x_0 \in \mathcal{C}^{\mathbb{R}}$, define a map $h : \mathcal{C}^{\mathbb{R}} \rightarrow \mathcal{C}^{\mathbb{R}}$ by $h(x) = x + x_0$, shifting the input path by x_0 . By pushing forward Wiener measure to the range $\mathcal{C}^{\mathbb{R}}$, we get a new measure $h_*w = wh^{-1}$ on $\mathcal{C}^{\mathbb{R}}$. Translational quasi-invariance of Wiener measure states that w and wh^{-1} are equivalent measures on $\mathcal{C}^{\mathbb{R}}$, so $w \ll wh^{-1}$ and $wh^{-1} \ll w$. In other words, both measures are absolutely continuous with respect to the other, implying that they have the same measure-zero sets.

The Cameron-Martin formula [2] is

$$\int_{\mathcal{C}^{\mathbb{R}}} f(z)dw(z) = e^{-\int_0^1 y'(t)^2 dt} \int_{\mathcal{C}^{\mathbb{R}}} f(x+y)e^{-2\int_0^1 y'(t)dx(t)}dw(x)$$

in which $f : \mathcal{C}^{\mathbb{R}} \rightarrow \mathbb{R}$ is a bounded functional for any z in a uniformly bounded set. This can be interpreted as the relatively mild condition that paths never get farther than some distance $M \geq 0$ from the origin. The path $y \in \mathcal{C}^{\mathbb{R}}$ is fixed with first derivative y' having bounded variation on $[0, 1]$. Rigorously, this means that

$$V_{y'}(0,1) = \sup \left\{ \sum_{j=1}^n |y'(t_j) - y'(t_{j-1})| : n \in \mathbb{N}, 0 = t_0 < t_1 < \dots < t_n = 1 \right\} < \infty.$$

In simpler terms, a function with bounded variation is one that does not oscillate too much. The path $x \in \mathcal{C}^{\mathbb{R}}$ in the Cameron-Martin formula is defined by the translation $z(t) = x(t) + y(t)$.

There is a useful theorem related to the Cameron-Martin formula that looks at subspaces $\mathcal{A}^{\mathbb{R}}$ of $\mathcal{C}^{\mathbb{R}}$ [2]. It says the following:

Theorem 1. *Given a path $y \in \mathcal{C}^{\mathbb{R}}$ with first derivative y' having bounded variation over $[0, 1]$ and a Wiener-measurable (w -measurable) subset $\mathcal{A}^{\mathbb{R}}$ of $\mathcal{C}^{\mathbb{R}}$, define $T\mathcal{A}^{\mathbb{R}}$ to be the translation of $\mathcal{A}^{\mathbb{R}}$ by y :*

$$T\mathcal{A}^{\mathbb{R}} = \{x \in \mathcal{C}^{\mathbb{R}} \mid x(t) \equiv z(t) - y(t), z \in \mathcal{A}^{\mathbb{R}}\}.$$

Then, we have

$$w(\mathcal{A}^{\mathbb{R}})e^{\int_0^1 y'(t)^2 dt} = \int_{T\mathcal{A}^{\mathbb{R}}} e^{-2\int_0^1 y'(t)dx(t)}dw(x).$$

3.2 Integration-by-parts formula

If a function f is Riemann-integrable with respect to α on the interval $[a, b]$, we have

$$\int_a^b f(x)d\alpha(x) + \int_a^b \alpha(x)df(x) = f(b)\alpha(b) - f(a)\alpha(a).$$

This is a generalization of the integration-by-parts formula to Riemann-Stieltjes integrals [1].

3.3 The Riemann-Stieltjes integral as the Riemann integral

If a function f is Riemann-integrable with respect to α on the interval $[a, b]$ and α has a continuous derivative α' on $[a, b]$, then

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx.$$

This gives us a condition to reduce the Riemann-Stieltjes integral to the traditional Riemann integral, where dx is Lebesgue measure [1].

4 Calculating the desired integral

In calculating the goal integral (Equation (4)), we must fix a path $y : [0, 1] \rightarrow \mathbb{R}$; we choose it to be C^2 and be such that $y(0) = y(1) = 0$, $y'(0) = y'(1) = 0$, and $y'' \neq 0$. We start by manipulating the Cameron-Martin

formula. Because $f \equiv 1$ is always bounded, we choose this function to input into the Cameron-Martin formula:

$$\int_{\mathcal{C}^{\mathbb{R}}} 1dw(z) = e^{-\int_0^1 y'(t)^2 dt} \int_{\mathcal{C}^{\mathbb{R}}} 1e^{-2\int_0^1 y'(t)dx(t)} dw(x).$$

Since $(\mathcal{C}^{\mathbb{R}}, \mathcal{B}_{\mathcal{C}^{\mathbb{R}}}, w)$ is a probability space (where $\mathcal{B}_{\mathcal{C}^{\mathbb{R}}}$ is the Borel σ -algebra of $\mathcal{C}^{\mathbb{R}}$), $w(\mathcal{C}^{\mathbb{R}}) = 1$, so the left-hand side becomes 1. We may then move the outer exponential on the right-hand side to the other side to get

$$e^{\int_0^1 y'(t)^2 dt} = \int_{\mathcal{C}^{\mathbb{R}}} e^{-2\int_0^1 y'(t)dx(t)} dw(x). \quad (5)$$

We want to use Equation (3) (Section 2.1) to rewrite the right-hand side of our integral; however, we must first verify that $e^{-2\int_0^1 y'(t)dx(t)} \in \mathcal{C}(\mathcal{C}^{\mathbb{R}})$.

Lemma 1. *The function $e^{-2\int_0^1 y'(t)dx(t)}$ is continuous on $\mathcal{C}^{\mathbb{R}}$, the space of continuous paths in \mathbb{R} beginning at 0.*

Proof. Since exponentials are continuous functions and compositions of continuous functions are continuous, it suffices to show that $\int_0^1 y'(t)dx(t) \in \mathcal{C}(\mathcal{C}^{\mathbb{R}})$. We want to show for every $\epsilon > 0$ that there exists a $\delta > 0$ such that

$$\|x_1 - x_2\|_{\infty} < \delta \implies \left| \int_0^1 y'(t)dx_1(t) - \int_0^1 y'(t)dx_2(t) \right| < \epsilon.$$

Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{\int_0^1 |y''(t)| dt}$. Since $y'' \not\equiv 0$ by assumption, $|y''(t)| > 0$ for some $t \in [0, 1]$, so $\int_0^1 |y''(t)| dt > 0$. Thus, our choice of δ makes sense. Assume that $\|x_1 - x_2\|_{\infty} < \delta$, so $\|x_1 - x_2\|_{\infty} < \frac{\epsilon}{\int_0^1 |y''(t)| dt}$.

$$\text{Now, consider the difference } \left| \int_0^1 y'(t)dx_1(t) - \int_0^1 y'(t)dx_2(t) \right|.$$

From the integration-by-parts formula for Riemann-Stieltjes integrals (Section 3.2), we obtain

$$\int_0^1 y'(t)dx_1(t) = y'(1)x_1(1) - y'(0)x_1(0) - \int_0^1 x_1(t)dy'(t)$$

and

$$\int_0^1 y'(t)dx_2(t) = y'(1)x_2(1) - y'(0)x_2(0) - \int_0^1 x_2(t)dy'(t).$$

Since we imposed the conditions on y that $y'(0) = y'(1) = 0$, we reduce these equations to

$$\int_0^1 y'(t)dx_1(t) = - \int_0^1 x_1(t)dy'(t)$$

and

$$\int_0^1 y'(t)dx_2(t) = - \int_0^1 x_2(t)dy'(t).$$

Substitution into our original difference expression yields

$$\left| - \int_0^1 x_1(t)dy'(t) + \int_0^1 x_2(t)dy'(t) \right| = \left| \int_0^1 (x_2 - x_1)(t)dy'(t) \right|.$$

We may then pull out the sup norm at the cost of an inequality, so

$$\left| \int_0^1 (x_2 - x_1)(t)dy'(t) \right| \leq \|x_1 - x_2\|_{\infty} \left| \int_0^1 dy'(t) \right|.$$

Because we chose $y \in C^2$, i.e., y' to be continuously differentiable, we may use Section 3.3 to replace the Riemann-Stieltjes integral with the classic Riemann integral:

$$\|x_1 - x_2\|_\infty \left| \int_0^1 dy'(t) \right| = \|x_1 - x_2\|_\infty \left| \int_0^1 y''(t) dt \right|.$$

We may pull the absolute value into the integral to get the inequality

$$\|x_1 - x_2\|_\infty \left| \int_0^1 y''(t) dt \right| \leq \|x_1 - x_2\|_\infty \int_0^1 |y''(t)| dt$$

and use our choice of δ , since $\|x_1 - x_2\|_\infty < \delta = \frac{\epsilon}{\int_0^1 |y''(t)| dt}$, to find that

$$\|x_1 - x_2\|_\infty \int_0^1 |y''(t)| dt < \frac{\epsilon}{\int_0^1 |y''(t)| dt} \int_0^1 |y''(t)| dt = \epsilon.$$

Therefore, we have that

$$\left| \int_0^1 y'(t) dx_1(t) - \int_0^1 y'(t) dx_2(t) \right| < \epsilon$$

implying that $\int_0^1 y'(t) dx(t) \in \mathcal{C}(\mathcal{C}^\mathbb{R})$, so $e^{-2 \int_0^1 y'(t) dx(t)} \in \mathcal{C}(\mathcal{C}^\mathbb{R})$. \square

Since $e^{-2 \int_0^1 y'(t) dx(t)} \in \mathcal{C}(\mathcal{C}^\mathbb{R})$, we can use the relation between integrals over $\mathcal{C}^\mathbb{R}$ and integrals over $\mathcal{C}_\theta^\mathbb{R}$ (Section 2.1; Equation (3)) to rewrite the right-hand side of Equation (5) as

$$\int_{\mathcal{C}^\mathbb{R}} e^{-2 \int_0^1 y'(t) dx(t)} dw(x) = \int_{\mathbb{R}} \int_{\mathcal{C}_\theta^\mathbb{R}} e^{-2 \int_0^1 y'(t) dx(t)} dw_\theta(x) d\theta.$$

Connecting equalities, Equation (5) then becomes

$$e^{\int_0^1 y'(t)^2 dt} = \int_{\mathbb{R}} \int_{\mathcal{C}_\theta^\mathbb{R}} e^{-2 \int_0^1 y'(t) dx(t)} dw_\theta(x) d\theta. \quad (6)$$

Notice the inner integral on the right-hand side is of the form of our integral of interest.

4.1 Stripping away the outer integral

The next goal is to remove the outer integral over \mathbb{R} in some way from Equation (6). This can be done using Theorem 1 on subspaces $\mathcal{A}^\mathbb{R}$ of $\mathcal{C}^\mathbb{R}$.

We consider the subspace $\mathcal{A}^\mathbb{R} \subseteq \mathcal{C}^\mathbb{R}$ in which paths end in the ray $(-\infty, a]$ for some fixed $a \in \mathbb{R}$, denoted $\mathcal{A}^\mathbb{R} = \mathcal{C}_{(-\infty, a]}^\mathbb{R}$. We use the same fixed y with the conditions as mentioned at the beginning of this Section (importantly, $y(0) = y(1) = 0$). Note that the translation of $\mathcal{C}_{(-\infty, a]}^\mathbb{R}$ by y , $T\mathcal{C}_{(-\infty, a]}^\mathbb{R}$, yields precisely our original subspace:

For any $x \in T\mathcal{C}_{(-\infty, a]}^\mathbb{R}$ and $z \in \mathcal{C}_{(-\infty, a]}^\mathbb{R}$, $x(0) \equiv z(0) - y(0) = 0 - 0 = 0$ and $x(1) \equiv z(1) - y(1) = z(1) \in (-\infty, a]$. Thus, any $x \in T\mathcal{C}_{(-\infty, a]}^\mathbb{R}$ will begin at 0 and end in $(-\infty, a]$. This implies that $x \in \mathcal{C}_{(-\infty, a]}^\mathbb{R}$, so $T\mathcal{C}_{(-\infty, a]}^\mathbb{R} = \mathcal{C}_{(-\infty, a]}^\mathbb{R}$ for our choice of y .

Theorem 1 and Equation (3) then imply

$$w(\mathcal{C}_{(-\infty, a]}^{\mathbb{R}}) e^{\int_0^1 y'(t)^2 dt} = \int_{\mathcal{C}_{(-\infty, a]}^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw(x) = \int_{-\infty}^a \int_{\mathcal{C}_{\theta}^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw_{\theta}(x) d\theta$$

for which $\theta \in (-\infty, a]$.

We can use our explicit definition of Wiener measure (Section 2.1; Equation (2)) with $A_1 = (-\infty, a]$ and $t_1 = 1$ to manipulate the left side of the equation, as $\mathcal{C}_{(-\infty, a]}^{\mathbb{R}} = \{x \in \mathcal{C}^{\mathbb{R}} \mid x(1) \in (-\infty, a]\}$:

$$e^{\int_0^1 y'(t)^2 dt} \int_{-\infty}^a u(x, 1) dx = \int_{-\infty}^a \int_{\mathcal{C}_{\theta}^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw_{\theta}(x) d\theta.$$

Replacing $u(x, 1)$ with its definition as a Gaussian (Section 2.1), we get

$$(2\pi)^{-1/2} e^{\int_0^1 y'(t)^2 dt} \int_{-\infty}^a e^{-x^2/2} dx = \int_{-\infty}^a \int_{\mathcal{C}_{\theta}^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw_{\theta}(x) d\theta.$$

We differentiate both sides with respect to a

$$\frac{d}{da} (2\pi)^{-1/2} e^{\int_0^1 y'(t)^2 dt} \int_{-\infty}^a e^{-x^2/2} dx = \frac{d}{da} \int_{-\infty}^a \int_{\mathcal{C}_{\theta}^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw_{\theta}(x) d\theta$$

and use the first part of the fundamental theorem of calculus to get

$$(2\pi)^{-1/2} e^{\int_0^1 y'(t)^2 dt} e^{-a^2/2} = \int_{\mathcal{C}_a^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw_a(x).$$

We can then apply a discrete integral (or sum) over the fiber $p^{-1}(\bar{a})$ of the covering map p for some endpoint $\bar{a} \in S^1$ (as $p^{-1}(\bar{a})$ is a countable set) to get a slight variation of our goal integral, Equation (4) from Section 2.2:

$$\int_{p^{-1}(\bar{a})} (2\pi)^{-1/2} e^{\int_0^1 y'(t)^2 dt} e^{-a^2/2} da = \int_{p^{-1}(\bar{a})} \int_{\mathcal{C}_a^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw_a(x) da.$$

In sum form, this looks like

$$\sum_{a \in p^{-1}(\bar{a})} (2\pi)^{-1/2} e^{-a^2/2} e^{\int_0^1 y'(t)^2 dt} = \int_{p^{-1}(\bar{a})} \int_{\mathcal{C}_a^{\mathbb{R}}} e^{-2 \int_0^1 y'(t) dx(t)} dw_a(x) da.$$

To eliminate the -2 constant in the right-hand exponential of the above equation, we can scale $y'(t)$, creating a new path $\hat{y}'(t) = -2y'(t)$. Note this scaling will not impact any of the conditions we imposed on y . Therefore, we rewrite our equation with our substituted $\hat{y}'(t)$ as

$$\sum_{a \in p^{-1}(\bar{a})} (2\pi)^{-1/2} e^{-a^2/2} e^{\frac{1}{4} \int_0^1 \hat{y}'(t)^2 dt} = \int_{p^{-1}(\bar{a})} \int_{\mathcal{C}_a^{\mathbb{R}}} e^{\int_0^1 \hat{y}'(t) dx(t)} dw_a(x) da.$$

Finally, we have a formula for calculating our desired integral:

$$\int_{\mathcal{C}_{\theta}^{S^1}} e^{\int_0^1 \hat{y}'(t) d\bar{x}(t)} d\mu_{\theta}^{S^1}(\bar{x}) = \int_{p^{-1}(\bar{\theta})} \int_{\mathcal{C}_{\theta}^{\mathbb{R}}} e^{\int_0^1 \hat{y}'(t) dx(t)} dw_{\theta}(x) d\theta = \sum_{\theta \in p^{-1}(\bar{\theta})} (2\pi)^{-1/2} e^{-\theta^2/2} e^{\frac{1}{4} \int_0^1 \hat{y}'(t)^2 dt}$$

or, directly,

$$\int_{\mathcal{C}_{\theta}^{S^1}} e^{\int_0^1 \hat{y}'(t) d\bar{x}(t)} d\mu_{\theta}^{S^1}(\bar{x}) = \sum_{\theta \in p^{-1}(\bar{\theta})} (2\pi)^{-1/2} e^{-\theta^2/2} e^{\frac{1}{4} \int_0^1 \hat{y}'(t)^2 dt}, \quad (7)$$

which holds for a fixed $y : [0, 1] \rightarrow \mathbb{R}$ with the following properties:

- $y \in C^2$
- $y(0) = y(1) = 0$
- $y'(0) = y'(1) = 0$
- y' has bounded variation on $[0, 1]$
- $y'' \neq 0$

References

- [1] T.M. Apostol. *Mathematical Analysis*. Addison-Wesley Series in Mathematics. Addison-Wesley, 1974. ISBN: 9780201002881.
- [2] R. H. Cameron and W. T. Martin. "Transformations of Weiner Integrals Under Translations". In: *Annals of Mathematics* 45.2 (1944), pp. 386–396. ISSN: 0003486X, 19398980. DOI: <https://doi.org/10.2307/1969276>.
- [3] I. B. Frenkel. "Orbital theory for affine Lie algebras". en. In: *Inventiones mathematicae* 77.2 (June 1984), pp. 301–352. ISSN: 1432-1297. DOI: 10.1007/BF01388449.
- [4] Terence Tao. *254A, Lecture 9: Ergodicity*. en. Feb. 2008. URL: <https://terrytao.wordpress.com/2008/02/04/254a-lecture-9-ergodicity/> (visited on 06/17/2024).